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THE

PHILOSOPHICAL REVIEW.

THE DOCTRINE OF SPACE AND TIME.

II. DIFFICULTIES CONNECTED WITH THE KANTIAN DOCTRINE OF SPACE.

MORE than two thousand years ago, it was argued by Zeno of Elea that motion is impossible, on the ground that, since space is infinitely divisible, no space, however small, can be passed over by a moving body. To go from one place to another, a body would have to pass through an unlimited number of intermediate spaces. That is, it would have to reach the last term of an unlimited series, which is absurd.

The more clearly this problem is stated, the more evident it seems to become that the difficulty is insurmountable. It appears to arise out of the very notion of space and of motion in space as continuous. "The idea expressed by that word *continuous*," says Professor Clifford,[1] "is one of extreme importance; it is the foundation of all exact science of things; and yet it is so very simple and elementary that it must have been almost the first clear idea that we got into our heads. It is only this: I cannot move this thing from one position to another, without making it go through an infinite number of intermediate positions. *Infinite;* it is a dreadful word, I know, until you find out that you are familiar with the thing which it expresses. In this place it means that between any two positions there is some intermediate position; between that and either of the others, again, there is some other intermediate; and so on *without any end.* Infinite means without any end. If you went on with that work of counting forever, you would never get any further than the beginning of it. At last you would only have two positions very close together, but not the same; and the whole process might be gone over again, beginning with those as many times as you like."

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[1] *Seeing and Thinking*, p. 134.

In this extract Professor Clifford plays directly into the hand of Zeno, although it is no part of his purpose to support the contention of that philosopher. He is merely trying to make quite clear what we mean by calling space continuous; and is it not generally admitted that space *is* continuous? But then how *can* anything move through space? The difficulties that beset a moving point Clifford has himself admirably exhibited, and again without the slightest intention of unduly emphasizing these difficulties or of denying the possibility of motion. He writes:[1]

"When a point moves, it moves along some line; and you may say that it traces out or describes the line. To look at something definite, let us take the point where this boundary of red on paper is cut by the surface of water. I move all about together. Now you know that between any two positions of the point there is an infinite number of intermediate positions. Where are they all? Why, clearly, in the line along which the point moved. That line is the place where all such points are to be found."

. . . "It seems a very natural thing to say that space is made up of points. I want you to examine very carefully what this means, and how far it is true. And let us first take the simplest case, and consider whether we may safely say that a line is made up of points. If you think of a very large number—say, a million—of points all in a row, the end ones being an inch apart, then this string of points is altogether a different thing from a line an inch long. For if you single out two points which are next one another, then there is no point of the series between them; but if you take two points on a line, however close together they may be, there is an infinite number of points between them. The two things are different *in kind*, not in degree."[2]

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[1] *Op. cit.*, pp. 143-4.

[2] *Ibid.*, pp. 146-7.

. . . "When a point moves along a line, we know that between any two positions of it there is an infinite number (in this new sense[1]) of intermediate positions. That is because the motion is continuous. Each of those positions is where the point was at some instant or other. Between the two end positions on the line, the point where the motion began and the point where it stopped, there is no point of the line which does not belong to that series. We have thus an infinite series of successive positions of a continuously moving point, and in that series are included all the points of a certain piece of line-room. May we say then that the line is made up of that infinite series of points ?

"Yes ; if we mean no more than that the series makes up the *points* of the line. But *no*, if we mean that the line is made up of those points in the same way that it is made up of a great many very small pieces of line. A point is not to be regarded as a *part* of a line, in any sense whatever. It is the boundary between two parts."[2]

Surely Zeno would have welcomed all this as directly establishing his position. "When a point moves along a line, we know that between any two portions of it there is an infinite number . . . of intermediate positions." "Infinite means without any end." The positions with which we are dealing are "the successive positions of a continuously moving point." Hence, to complete its motion over any given line whatever, the moving point *must* pass, one by one, an *endless* series of positions, and must finish with the *end* position. If the moral of this is not that a point cannot move along a line, there is no validity in human reasonings.

Again: The moving point must take, one by one, the "successive positions" in the series. Even the (conscious or unconscious) Kantian has his preference in absurdities, and rejects some rather than others. Clifford does not conceive the point as in two positions at once, or as making some ingenious flank movement by means of which it can 'scoop in' a whole stretch of line simultaneously. It must move *along* the line, from end to end, taking one position at a time, and taking them in their order. It cannot make jumps, and are not the positions "successive"? Its path seems clearly marked out for it—a smooth road, and without turnings. Alas! the line is "continuous"! The point cannot take *successive* positions, for have we not seen that no position can immediately succeed any other on a continuous line? "Between any two positions there is some intermediate position; between that and either of the others, again, there is some other intermediate; and so on *without any end.*" Can any living soul conceive the gait that must be adopted by a point, which must move continuously (without jumps?) over a line, and yet is debarred from passing from any one position to the next in the series? It cannot pass first to some position which is not the next, and then get around to the next after a while. That is palpably absurd. And it cannot pass to the next at once, for there is no next. I can imagine the shade of Zeno rubbing its hands over this development of his doctrine. "The way for a point to get on," says Clifford, "is for it never to take the next step." "Of course that means," adds Zeno with ghostly laughter, "that a point cannot get on at all."

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[1] Professor Clifford has used the word 'number' in two senses, a quantitative and a qualitative. By number in the latter sense he means simply 'unlimited units.'

[2] *Op. cit.*, pp. 149–50.

And what shall we say to the statement that although "all the points of a certain piece of line-room" are included in the "infinite series of successive (*sic*) positions of a continuously moving point," yet the line is not made up of these points, but is made up "of a great many very small pieces of line"? What are these small pieces of line, which are to be distinguished from the whole series of points? They are not material things, for we are not now discussing a bit of string or a chalk-mark, but we are discussing a geometrical line, an aspect of space. What lies between any two points on the line? More points, for one thing. What else? Bits of line. But what are bits of line? When a point has moved over a line, has it done anything but pass through a series of successive positions? It seems reasonable, at first sight, to assume that such a series of positions is what we mean by a line. We are informed, however, that a point is not to be regarded as *part* of a line in any sense whatever. It is "the boundary between two parts." Does the assumption of these bits of line, which are not positions, but lie between positions, make more comprehensible the motion of a point over a line?

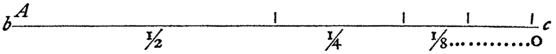
Manifestly not. If the bits of line could be supposed to take up some of the line-room in such a way as to reduce the number of points, they might be of some help, but no one supposes them to do this. Bits of line or no bits of line, the moving point must occupy successively all the positions in an infinite series. And if we turn our attention from the points, and confine it to the bits of line, we are no better off. If the number of points is endless, so is the number of bits of line, for these separate the points, which are only their boundaries, and we are forced to ask ourselves how an endless series of bits of line can come to an end in a last bit which completes the line. It is not a whit easier to conceive of a given finite line as composed of bits of line, than it is to conceive of it as composed of points, if we once admit that the line in question is infinitely divisible. We have only added a new element of mystification. What do we *mean* by these mysterious bits of line? Has the point which is passing over a series of positions anything whatever to do with them? Do they really separate the positions, so that they must be *jumped* in getting along the series, or does the point, after all, meet nothing but *positions*, never that which separates them?

The attempt is sometimes made to avoid the difficulty of assuming that a point moving over a line can progressively exhaust an infinite series, by laying much emphasis upon the fact that the members of the series are exceedingly small, and can be passed over with great rapidity. Infinitesimal spaces, it is argued, are passed over in infinitesimal times, and all these infinitesimals are included in the finite space and time of the motion. But it must be evident to anyone capable of the least clearness of thought that quibbling over the size of the members of the series, in the case either of space or of time, is wholly wide of the mark. Whether things are big or little, if the supply of them be truly endless, one can never get to the end of the supply. The rapidity with which the terms of the series are exhausted has obviously no effect in facilitating an approach to that which cannot, by hypothesis, exist, *i. e.*, to a final term. The proposed solution of the problem rests upon the implicit assumption that, provided only things are small enough, it is legitimate to reason about them in an incoherent way, and to make self-contradictory statements. I know of no way in which this assumption can be defended, unless it be by claiming that it is an 'intuition.'

If, then, in order to move a body, I must reach the end of an endless series, I may reasonably conclude that I cannot move a body. This is as clear as it is possible for anything to be. No exception can be taken to Zeno's argument, if the assumption upon which it rests be once granted. One is not at liberty to admit that there are difficulties connected with the statement that a point can move along an infinitely divisible line, and to hold, in spite of these difficulties, that the statement should be approved as being the least objectionable that can be made touching the subject. One should bear in mind that this amounts to saying that what is flatly self-contradictory and, hence, intrinsically absurd, is at least less objectionable, as an article of faith, than is something else. I wish to emphasize the fact that no opposing doctrine, try as it may, can possibly be worse. At best it can only succeed in being as bad.

The difficulties arising out of the doctrine of the infinite divisibility of finite spaces have been so long before the philosophic public that it is tired of them, and its sense has grown deadened to their significance. They are recognized; they arouse a fugitive interest; they are made to yield a favorable occasion for a pleasing exercise of the ingenuity, and then they are put back again into their box and their existence is ignored. They are not taken seriously, and the serious interest with which the ancients approached them is even characterized as pathologic. But whether we face them or not, the difficulties are there just the same. They do not become non-existent merely because they are overlooked; and it is surely a crying disgrace to human reason that a theory of the nature of space should complacently be accepted as truth, which admittedly runs into unresolved self-contradictions. So important is it that the reader should clearly realize what is implied in the Kantian doctrine, that I will beg his indulgence while I set forth a rather interesting bit of reasoning, the sole defect in which is that it rests upon the assumption contained in that doctrine. It is, in all other respects, beyond criticism.

Let us suppose a point *A* moving uniformly over a finite line *bc*, at such a rate that it will complete the distance in one second.



Since the motion is uniform, the point will pass over one half of the line in half a second; it will pass over one half of the remainder, or one-fourth of the line, in a quarter of a second, etc. When the point has passed over the whole line, it will have completed the descending series: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ . . . . 0.

We may set aside for the present purpose the 'difficulties' connected with the point's getting a start along an infinitely divisible line, and with the completion of an endless series in general. We will accept it as a fact that the line is infinitely divisible and can be passed over, in an infinitely divisible second, by a point moving at a uniform rate. All these are good Kantian assumptions. It seems to follow rigorously that both the line and the second are exhausted as our descending series indicates, and that both come to an end only when the series is terminated. The motion can be completed; the second can be completed; the series can be completed. In fact, all three *are* completed simultaneously. In the case, then, of a point moving uniformly over a finite line, we have evidence of the fact that an infinite descending series, such as $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ . . . . 0, can be, and is, completed.

Now let us suppose a circular disc set revolving around its center, in the plane of this paper, in such a manner that, at the first revolution, a point *P* on its circumference is carried around to the place at which it was before in half a second, at the second revolution, in a quarter of a second, at the third, in an eighth of a second, etc. It is clear that at the end of one second from the beginning of the motion the disc will be revolving with infinite rapidity, or, in other words, the time of *P*'s revolution will be reduced from half a second to zero. We have here a descending series of exactly the same kind as the one we had above; the times taken up by the successive revolutions are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ . . . . 0.

Thus, when the disc is revolving with infinite rapidity, there is no time at all between *P*'s leaving the place at which it was and coming back to it again, which means, if it means anything, that *P* is *always* at the same place. But, since similar reasoning will apply to any other position through which *P* is supposed to pass in each of its revolutions (for the interval between its leaving that position and returning to it again is reduced to zero by the completion of the series), we can prove just as cogently that *P* is in the whole series of positions all the time. We can prove, in other words, that when the disc revolves with infinite rapidity, *P* is always all around the disc at once.

I suggest this argument to those who incline to the at present rather unfashionable scholastic notion that the whole soul is simultaneously in all parts of the body—*tota in toto et tota in utraque parte.* It may be used as a new weapon of defense, and has the advantage of being based upon principles admitted by their antagonists. If there be any truth in the Kantian doctrine of the infinite divisibility of space and time, why should not the soul be thus ubiquitous? It has only to move fast enough and it may succeed in being everywhere at once. The trick is simple—let it reduce to zero the time between its setting out from a given spot and its getting around to it again. It will, then, never be away from that spot, and it will also always be at every other spot in the line of its vibration. To those who find repugnant the thought of this midge's dance of the soul through all parts of the body, I suggest that there is nothing in this doctrine to prevent one from believing that through it all the soul retains the quiet seat in the pineal gland assigned it by Descartes. There it remains, like a spider at the center of its web; and one can rest one's mind by thus conceiving it. On the other hand, in those heroic moods in which the philosopher loves to emphasize the magic powers which distinguish mind from matter, independence of space and what not, one can reflect upon the storm and stress of its inconceivable motion—a motion which appears to resemble rest, and yet is its extremest opposite; a motion which consists in being at rest in every place and in no place simultaneously. Then one can proudly maintain that, though the soul be in the pineal gland, it is not imprisoned there, like an impotent lump of matter, hemmed in by the walls of its cell, and unable to break through them. It is there, as it is everywhere, by its own tireless energy—there and not there, there and everywhere, a standing miracle, a living contradiction.

The topic is one upon which an enthusiast might dilate; but even enthusiasm should not be allowed to run into injustice, and the mention of matter reminds me that, for the Kantian, matter, too, may have its magical properties. We began with a revolving disc, and found that a point upon its circumference may be, under certain conditions, all around the disc at once. But if this be so, it must be possible for a material particle in the tire of a revolving wheel to be all around the wheel at once, when the wheel is revolving with infinite rapidity, and, thus, to occupy the same space with all the other particles in its path. Is this a new insight into the constitution of matter? Shall we say that every particle of matter excludes from the space it occupies every other particle when, and only when, its motion is not too rapid? Or shall we say that, although it *is* conceivable that an infinite series may be completed by a point moving along a line, yet it *is not* conceivable that an infinite series can be completed by the revolutions of a disc? Is it an 'intuition' that there is this difference between moving points and revolving discs?

But, it is objected, all this is sheer nonsense; no point can possibly be in more than one position at one time, nor is it possible that a point should move so rapidly as always to remain in the same spot. I answer: Of course it is sheer nonsense; but I insist that *the whole nonsensical edifice rests upon the one nonsensical assumption that an endless series can be completed by a progress which results in the attainment of a final term.* This is the assumption to which his peculiar views of the infinite divisibility of space and time force the Kantian. Grant this assumption and the rest follows of itself. The reasoning contains no other error. Its steps, briefly stated, are as follows:

1. If finite spaces and times are infinitely divisible, a point moving uniformly over a finite line *must* be able to pass through an endless series of positions and arrive at the very end.

2. The total space and time of the motion may be so divided as to be truly represented by the descending series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots 0$.

3. If it is possible for one such series to be completed, there is absolutely no reason for affirming that another series of exactly the same kind may not be.

4. Hence, if it is conceivable that a disc may complete one revolution upon its center in half a second, the next in a quarter of a second, etc., etc., there is no reason for affirming that it is theoretically impossible for it to attain such a rate of speed that the time of its revolution will be reduced to zero.

5. When it is thus reduced to zero, it is clear that there is no time whatever during which a point upon the circumference of the disc is away from the position in which it was at the beginning of the motion, etc.

The conclusions of this reasoning are highly unpalatable; but there is only one way to avoid them, and that is to repudiate the foundations upon which they rest. Perhaps I should amend this statement by saying there is only one *logical* way to avoid them. Practically, of course, we can avoid them by turning our minds from the whole subject, and this is what is commonly done. The unpleasant consequences of philosophic reasonings may be put to rout by an enemy who has not borrowed his arms from Aristotle or from his successors. "I dine," writes Hume,[1] "I play a game of backgammon, I converse, and am merry with my friends; and when, after three or four hour's amusement, I would return to these speculations, they appear so cold, and strained, and ridiculous, that I cannot find it in my heart to enter into them any further." In such a mood logical difficulties are not taken seriously, and the mind drifts upon the stream of its habitual associations. It is worthy of remark that such moods are by no means exclusively the result of relaxation and conviviality. An attachment to the doctrines of this or of that school of thought, doctrines to which we have grown accustomed, and which seem to place at least some sort of ground under our feet; the agreeable sense that we belong to a party, and are not groping our way alone in the maze of speculations which confronts the philosopher; these things, and such as these, may disincline us to take seriously even the most serious of difficulties. We choose to jolt our way along upon the old road, even over an occasional self-contradiction. It seems better than to seek a smoother track, which is little frequented, and which may, for all we know, lead anywhere or nowhere. Accordingly, we take up an exposition of the inconsistencies which arise out of the Kantian doctrine, read it through, indulgently compliment the author upon his 'acuteness,' and, feeling unable to point out any actual flaw in his argument, we take our stand upon what may be called the platform of the liberal-conservative in philosophy, saying: 'There are undoubtedly difficulties connected with the doctrine of the infinite divisibility of finite spaces, but the way to avoid these difficulties is not to repudiate what is undoubted truth, and to take refuge in a shallow empiricism,' etc. Although the occasioning cause may be different, our attitude of mind is distinctly Humian.

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[1] *Treatise of Human Nature*, Book I, Part IV, § 7.

Before closing this discussion of the Kantian doctrine of space, I must comment briefly upon one attempt to avoid the enormities we have been passing in review, which does not repudiate the doctrine of the infinite divisibility of finite spaces, and which yet does not simply avert its eyes from the painful consequences of the doctrine. This attempt consists in maintaining that we are not bound to hold that every finite space consists of an infinite number of finite spaces, for space is *infinitely divisible*, not *infinitely divided.* This quibble—for, although it has a venerable history, it is nothing more—need not detain us very long. We have only to ask how it helps us in the case of the moving point. The line over which the point has moved is infinitely divisible. What does this mean? We call a line divisible, because we believe that it can be divided; and we believe that it can de divided (theoretically of course), because it is composed of parts. If we did not believe it to be composed of parts, we should not regard it as divisible. By saying that the line is infinitely divisible, we mean simply that it is composed, not of a limited, but of an unlimited number of parts; and by saying that the motion of a point over it is continuous, we mean that the point must take successively an infinite series of positions. Now our point has completed its progress; it is at the end of the line. Has it, or has it not, passed over every part of the line? Has it, or has it not, been successively in an endless series of positions? It is trivial to raise the question whether the parts of the line, the positions along it, have been counted or not. If the line is infinitely divisible, and if the point moves along it, it evidently comes to the end of an endless series at every step of its progress.

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